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Stability Criteria for a General Class of Finite Difference Schemes

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The von Neumann stability analysis is performed on a generalized class of two-level space-centered finite difference schemes known as Lerat-Peyret schemes. In the present study, these schemes are used to solve the convection-diffusion equation in an effort to obtain a better assessment of the accuracy of such schemes. Exact stability criteria are derived for each of five different schemes applied to the Burgers equation. Exact results for the amplification factor and phase error are plotted for several values of Courant number and diffusion parameter for each scheme. These exact results, which are presented here for the first time, are extremely valuable in assessing the dissipation and dispersion characteristics of each scheme. This study shows promise for extending this analysis to the Navier-Stokes equations.

Nomenclature

A	= Jacobian of inviscid flux function
B	= Jacobian of viscous flux function
c	= convection speed
E	= total energy
F	= inviscid fluxes in x direction
$\mathfrak{F}$	= one-dimensional inviscid flux function
k	= coefficient of thermal conductivity
p	= pressure
Q	= vector of dependent variables
q	= Courant number
R	= gas constant
r	= diffusion parameter
T	= temperature
U	= viscous fluxes in x direction
u	= velocity components in x direction
$\tilde{u}$	= predictor value
α, β, γ	= scheme parameters
$\bar{\gamma}$	= ratio of specific heats
Δt	= time step
Δx	= spatial mesh size in x direction
λ	= wave number
μ	= coefficient of dynamic viscosity
ν	= kinematic viscosity
ξ	= phase angle
ρ	= density, amplification factor
τ	= components of stress tensor
ϕ	= relative phase error

I. Introduction

OVER the past few years, significant increases in computing speed and memory have driven applications of computational fluid dynamics methods toward problems involving increasingly complex flow phenomena. These complex flows include unsteady features such as separation, vortex shedding and interaction, acoustics, chemical reactions, and turbulent phenomena. Any hope of analyzing these types of flows relies heavily on accurate resolution of a wide range of temporal and spatial scales present in the physics. To ensure that the required resolution is achieved, it is essential to evaluate the accuracy and stability of the numerical methods to be used in the solution of such flow problems. There are several different methods for analyzing the stability of a given scheme for simple model equations, including the von Neumann method, the modified equation approach,[1] and heuristic stability theory.[2] Several researchers have used such methods to conduct detailed stability studies[3-5]; however, in practical problems many efforts have relied on empirical approaches because of the complexity involved.

In the present work, analytical expressions for stability criteria are developed for a class of two-step finite difference schemes. Analytical expressions are derived for the amplification factor and phase error for five different schemes, and plots of the results are compared with those for the approximate expressions. These expressions are obtained for both the inviscid and viscous Burgers equations to validate the analyses. The exact analytic expressions for the various schemes for the viscous Burgers equation are believed to be presented here for the first time. In addition, specific stability criteria are given for each scheme. The analyses developed in this effort[6] provide a critical step in obtaining rigorous stability criteria for various schemes applied to the Navier-Stokes equations. These results provide significant insight into the role of spatial and temporal step size in resolving critical features of complex flows and in optimizing computational procedures to resolve required time and length scales.

II. Convection-Diffusion Equation and Its Numerical Approximation

Convection (or advection) and diffusion (or conduction) are two of the most common transport phenomena in physics. The differential equation that describes these two phenomena, known as the convection-diffusion equation, is a mixed hyperbolic-parabolic partial differential equation. In fluid dynamics the convection and diffusion transport processes are governed by the set of partially parabolic partial differential equations known as the Navier-Stokes equations. To evaluate the accuracy of the numerical simulation procedure one should perform a stability analysis on any difference scheme applied to the Navier-Stokes equations. In this study, as a first step, a simple model equation, the Burgers equation, is taken as an illustrative example of the Navier-Stokes equations, since it contains both convection and diffusion terms. It can be seen that, at least for the one-dimensional case, when they are written in nonconservative form, the Navier-Stokes equations have exactly the same form as the one-dimensional Burgers equation.

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A. One-Dimensional Navier-Stokes Equations

For simplicity, in this study, the one-dimensional Navier-Stokes equations are considered. The multidimensional cases are left for further investigation.

For the one-dimensional case, we have

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = \frac{\partial U}{\partial x} \tag{1}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} Q_2 \\ \dfrac{Q_2^2}{Q_1} + p \\ (Q_3 + p)\dfrac{Q_2}{Q_1} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} + k\dfrac{\partial T}{\partial x} \end{bmatrix}$$

and

$$\tau_{xx} = \frac{4}{3}\mu \frac{\partial u}{\partial x}$$

$$p = (\bar{\gamma} - 1)(E - \frac{1}{2}\rho u^2)$$

$$T = \frac{\bar{\gamma} - 1}{\rho R}(E - \frac{1}{2}\rho u^2)$$

Equation (1) can be reduced to the form[6]

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = B\frac{\partial^2 Q}{\partial x^2} + C \tag{2}$$

which can be verified by performing the indicated matrix multiplication.

For $C = 0$, the one-dimensional Navier-Stokes equation becomes

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = B\frac{\partial^2 Q}{\partial x^2} \tag{3}$$

B. Illustrative Example: Burgers Equation

To illustrate the von Neumann stability analysis,[7] let us first consider a simple case using the one-dimensional Burgers equation, which has the same form as the one-dimensional Navier-Stokes equation without the source term, Eq. (3).

Taking the Burgers equation as the model equation has two advantages here. First, there exist several exact (analytic) solutions for the Burgers equation with which the numerical results can be compared. Second, because the Burgers equation has the same form as the one-dimensional Navier-Stokes equation, the stability analysis for the Burgers equation will be a guide in the derivation and procedure of the stability analysis for the one-dimensional Navier-Stokes equation.

For the one-dimensional model equation, consider Eq. (3) to be of the form

$$\frac{\partial u}{\partial t} + \frac{\partial \mathfrak{F}(u)}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \tag{4}$$

If $\mathfrak{F}(u) = ½u^2$, Eq. (4) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{5}$$

which is the one-dimensional nonlinear viscous Burgers equation.

If $\mathfrak{F}(u) = cu$, Eq. (4) becomes

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{6}$$

which is the one-dimensional linear viscous Burgers equation.

When $\nu = 0$, the left-hand side vanishes, and Eqs. (5) and (6) become the inviscid Burgers equations.

C. Difference Schemes

Two types of difference schemes that have been widely used are space-centered schemes and upwind difference schemes. In this study, only space-centered schemes are investigated.

In 1973, Lerat and Peyret proposed a generalization of two-level predictor-corrector-type schemes for the model equation, Eq. (2),[8] which is in predictor-corrector form. Applying predictor and corrector expressions to the linear Burgers equation, Eq. (6), in which c and ν are constants, we have the following:

Predictor:

$$\begin{aligned}\bar{u}_j = {} & (1-\beta)u_j^n + \beta u_{j+1}^n - \alpha q(u_{j+1}^n - u_j^n) + \alpha r[\gamma(u_{j+2}^n \\ & - 2u_{j+1}^n + u_j^n) + (1-\gamma)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)]\end{aligned} \tag{7}$$

Corrector:

$$\begin{aligned}u_j^{n+1} = {} & u_j^n - (q/2\alpha)[(\alpha - \beta)u_{j+1}^n + (2\beta - 1)u_j^n \\ & + (1 - \alpha - \beta)u_{j-1}^n + \bar{u}_j - \bar{u}_{j-1}] \\ & + (r/2\alpha)[(2\alpha - 1)(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \\ & + (1-\beta)(\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1}) \\ & + \beta(\bar{u}_j - 2\bar{u}_{j-1} + \bar{u}_{j-2})]\end{aligned} \tag{8}$$

where $q = c\Delta t/\Delta x$ is the Courant number, and $r = \nu\Delta t/\Delta x^2$ is the diffusion parameter. Choosing different values of α, β, and γ in Eqs. (7) and (8) gives several schemes that have been investigated for various applications.

III. Analyses and Procedure

A. Derivation

After substitution and considerable algebra,[6] the expression for u_j^{n+1} in Eq. (8) becomes

$$\begin{aligned}u_j^{n+1} = {} & Au_{j-3}^n + Bu_{j-2}^n + Cu_{j-1}^n + Du_j^n \\ & + Eu_{j+1}^n + Fu_{j+2}^n + Gu_{j+3}^n\end{aligned} \tag{9}$$

where

$$A = ½\beta(1-\gamma)r^2$$

$$\begin{aligned}B = {} & r/2\{\beta(1-\beta)/\alpha + q(1+\beta-\gamma) \\ & + r[\beta(3\gamma - 2) + (1-\gamma)(1-3\beta)]\}\end{aligned}$$

$$\begin{aligned}C = {} & ½\{q(q+1) + (r/\alpha)[(2\alpha - 1) + (2\beta - 1)^2] \\ & + 2rq[2(\gamma - \beta) - 1] + r^2(10\beta - 15\beta\gamma + 5\gamma - 4)\}\end{aligned}$$

$$\begin{aligned}D = {} & 1 - q^2 + (r/\alpha)[3\beta(1-\beta) + 2\alpha] + 3rq(\beta - \gamma) \\ & + ½r^2[5(2\beta - 1)(2\gamma - 1) + 1]\end{aligned}$$

$$E = \tfrac{1}{2}\{q(q-1) + (r/\alpha)[(2\alpha-1) + (2\beta-1)^2] + 2rq[2(\gamma-\beta)+1] + r^2(5\beta - 15\beta\gamma + 10\gamma - 4)\}$$

$$F = r/2\{\beta(1-\beta)/\alpha - q(1-\beta+\gamma) + r[\gamma(3\beta-2) + (1-\beta)(1-3\gamma)]\}$$

$$G = \tfrac{1}{2}\gamma(1-\beta)r^2$$

Now substituting a discrete Fourier mode into Eq. (9), and using the relationship $e^{im\xi} = \cos(m\xi) + i\ \sin(m\xi)$, where $\xi = -\lambda\Delta x$ and λ is the wave number, we have

$$\begin{aligned}\rho(\xi) &= Ae^{i3\xi} + Be^{i2\xi} + Ce^{i\xi} + D + Ee^{-i\xi} + Fe^{-i2\xi} + Ge^{-i3\xi} \\ &= D + (A+G)\cos(3\xi) + (B+F)\cos(2\xi) + (C+E)\cos(\xi) \\ &\quad + i[(A-G)\sin(3\xi) + (B-F)\sin(2\xi) + (C-E)\sin(\xi)] \qquad (10)\end{aligned}$$

then the modulus of $\rho(\xi)$ is

$$|\rho(\xi)|^2 = \{\mathrm{Re}[\rho(\xi)]\}^2 + \{\mathrm{Im}[\rho(\xi)]\}^2 \qquad (11)$$

giving the amplification factor that is defined as $|\rho(\xi)|$.

For the phase error, we have

$$\phi\lambda\Delta t = \tan^{-1}\left\{\frac{\mathrm{Im}[\rho(\xi)]}{\mathrm{Re}[\rho(\xi)]}\right\}$$

but $-\pi/2 < \tan^{-1}(z) < \pi/2$, and $0 \le \phi\lambda\Delta t \le \pi$; therefore,

$$\phi = \begin{cases} \dfrac{c}{(-q\xi)}\tan^{-1}\left\{\dfrac{\mathrm{Im}[\rho(\xi)]}{\mathrm{Re}[\rho(\xi)]}\right\}, & \mathrm{Re}[\rho(\xi)] \ge 0 \\ \dfrac{c}{(-q\xi)}\pi - \dfrac{c}{(-q\xi)}\left|\tan^{-1}\left\{\dfrac{\mathrm{Im}[\rho(\xi)]}{\mathrm{Re}[\rho(\xi)]}\right\}\right|, & \mathrm{Re}[\rho(\xi)] < 0 \end{cases} \qquad (12)$$

where ϕ is the relative phase error that has non-negative values, and the ϕ graph will be in the upper half-plane.

B. Exact Formula

Substituting $A, B, \ldots, G$ into the previous equation for the amplification factor, we have

$$|\rho(\xi)|^2 = a^2[q - 2bqr + 2b^2(\beta-\gamma)r^2]^2 + \left\{1 - bq^2 + 2b^2(\beta-\gamma)qr + 2b\left[\frac{\beta(1-\beta)}{\alpha}b - 1\right]r + 2b^2[1 - b(\beta+\gamma-2\beta\gamma)]r^2\right\}^2 \qquad (13)$$

where $a = \sin(\xi)$, and $b = 1 - \cos(\xi)$.

This exact analytical expression for the amplification factor is believed to be derived here for the first time.

The necessary and sufficient condition (for periodic boundary conditions) for stability is

$$|\rho(\xi)| \le 1$$

C. Approximate Formulas at ξ Near Zero

The approximations for the amplification factor and phase error at ξ near zero show the properties of dissipation and dispersion, respectively.

Using Taylor series expansions for the sine and cosine functions for ξ near zero, neglecting terms of $O(\xi^6)$, we will get

$$|\rho(\xi)|^2 = 1 - 2r\xi^2 + \left\{\frac{1}{4}q^2(q^2-1) + rq(\beta-\gamma) - rq^2 + r\left[\frac{1}{6} + \frac{\beta(1-\beta)}{\alpha}\right] + 2r^2\right\}\xi^4$$

For small z, $\sqrt{1+z} = 1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \mathcal{O}(z^3)$, where $z = \xi^2$, and so

$$|\rho(\xi)| = 1 - r\xi^2 + \Psi_0\xi^4 \qquad (14)$$

where

$$\Psi_0 = \frac{1}{2}\left\{\frac{1}{4}q^2(q^2-1) + rq(\beta-\gamma) - rq^2 + r\left[\frac{1}{6} + \frac{\beta(1-\beta)}{\alpha}\right] + r^2\right\}$$

The dissipation is determined by the even powers of ξ in Eq. (14), which will be discussed further in Sec. IV.

For the phase error, from Eq. (12), repeating the Taylor series expansion procedure, and for small z, $[1/(1-z)] = 1 + z + z^2 + \mathcal{O}(z^3)$, $\tan^{-1}z = z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \mathcal{O}(z^7)$, and so

$$\phi = -c[1 - \tfrac{1}{6}(1-q^2)\xi^2 + \Gamma_0\xi^4] \qquad (15)$$

where

$$\Gamma_0 = \frac{1}{120} + \frac{1}{24}q^2 - \frac{1}{20}q^4 + \frac{1}{2}\left[\frac{\beta(\beta-1)}{\alpha}r - r^2 + (\gamma-\beta)\left(rq - \frac{r^2}{q}\right) + rq^2\right]$$

For the inviscid Burgers equation, $r = 0$, and so

$$|\rho(\xi)| = 1 - \tfrac{1}{8}q^2(1-q^2)\xi^4$$

$$\phi = -c\left[1 - \frac{1}{6}(1-q^2)\xi^2 + \left(\frac{1}{120} + \frac{1}{24}q^2 - \frac{1}{20}q^4\right)\xi^4\right]$$

The dispersion is governed by the magnitude of ϕ. This too will be discussed later.

IV. Results and Discussion

A. Results

The exact and approximate formulas for the amplification factor and phase error obtained in the preceding section yield many useful and interesting observations as shown in the following sections.

1. *Amplification Factor and Phase Error for Selected Schemes*

Using Eqs. (10) and (12), a computer program was developed to calculate the exact amplification factor and phase error. The data for the calculation are given in Table 1.

The graphs of the amplification factor and phase error for the inviscid Burgers equation are identical for each scheme and with the results in Ref. 9. This also verifies that, for the linear inviscid Burgers equation, all second-order space-centered schemes are identical with the Lax-Wendroff scheme.[10]

For the viscous Burgers equation, the graphs for different values of q and r are shown in Figs. 1–8 for the five different numerical schemes given in Table 1.

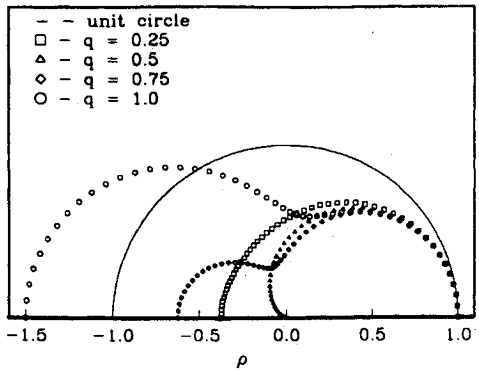
2. *Approximations for Small* ξ

The approximations of the amplification factor and phase error for the inviscid and viscous Burgers equations are summarized in Table 2.

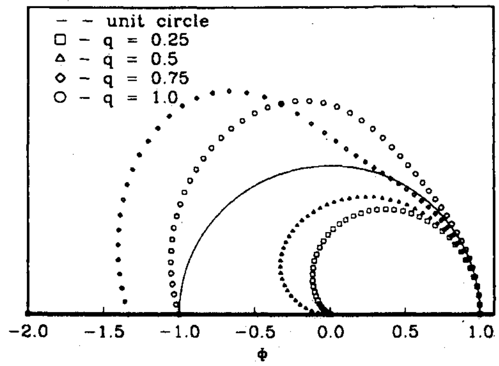
Table 1 Data for calculation of amplification factor and phase error

$q = c(\Delta t/\Delta x)$	——	0.25	0.5	0.75	1.0
$r = \nu(\Delta t/\Delta x^2)$	0.0	0.25	0.5	0.75	1.0

Scheme	α	β	γ
MacCormack (FB)	1.0	0.0	0.0
MacCormack (BF)	1.0	1.0	1.0
Lax-Wendroff	0.5	0.5	0.5
Rubin-Burstein	1.0	0.5	0.5
Peyret-Taylor	$1 + \sqrt{5}/2$	0.5	0.5

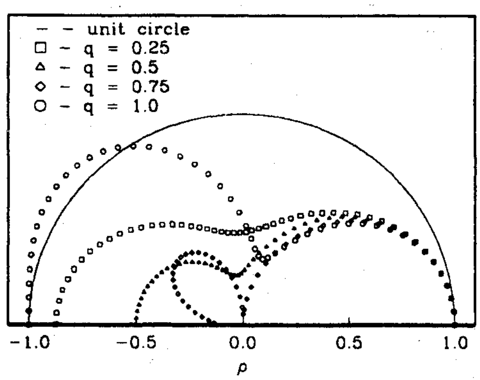


a) **Amplification Factor**

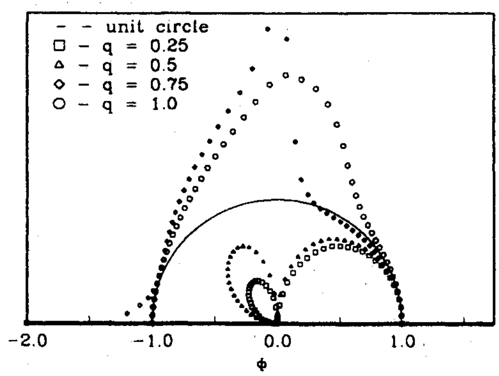


b) **Phase Error**

Fig. 1 Amplification factor and phase error for MacCormack (FB and BF) scheme ($r = 0.25$).



a) **Amplification Factor**



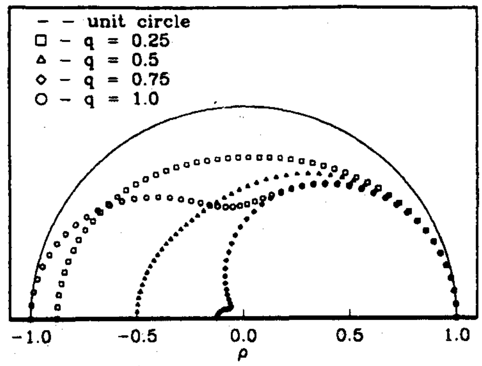
b) **Phase Error**

Fig. 2 Amplification factor and phase error for MacCormack (FB and BF) scheme ($r = 0.5$).

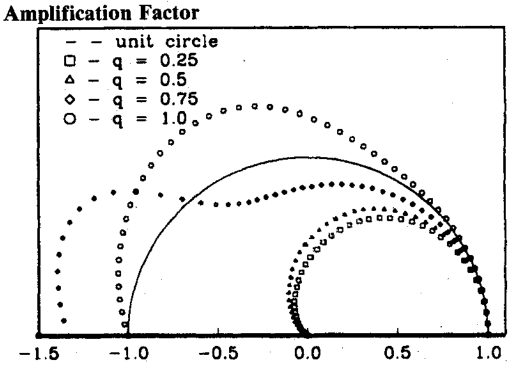
3. Dissipation and Dispersion

The smearing of physical discontinuities due to numerical dissipation and the nonphysical oscillations produced by numerical dispersion are of critical importance in evaluating the accuracy of a numerical scheme for resolving complex flow features.

From Eq. (14), it can be seen that the lowest power of ξ is 4; for the inviscid Burgers equation, the Lerat-Peyret generalized scheme is dissipative of order 4; for the viscous Burgers equation, noting the lowest power of ξ, the scheme becomes dissipative of order 2. This is due to the physical dissipation that occurs naturally in the viscous terms of the viscous Burgers equation.

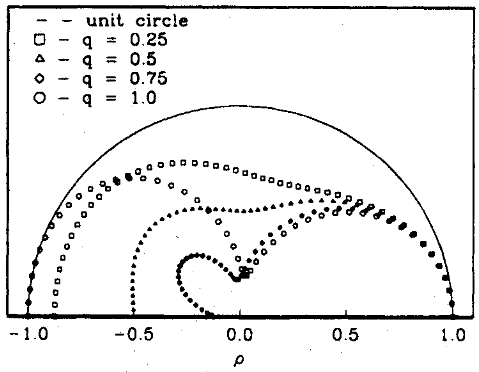


a) **Amplification Factor**

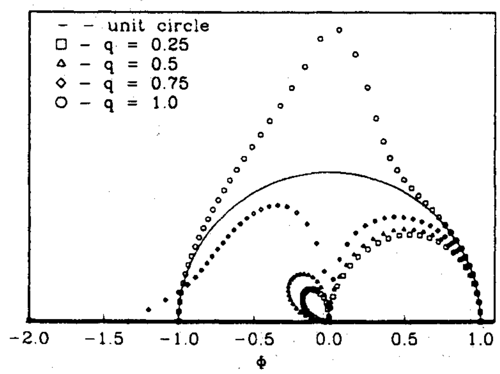


b) **Phase Error**

Fig. 3 Amplification factor and phase error for Lax-Wendroff scheme ($r = 0.25$).



a) **Amplification Factor**

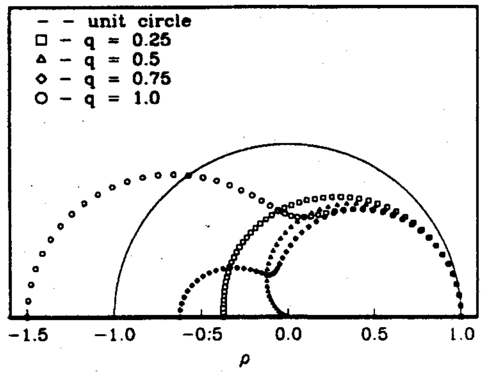


b) **Phase Error**

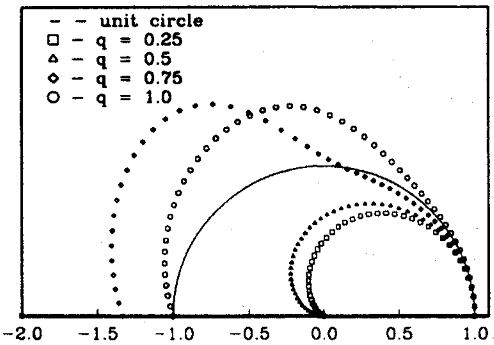
Fig. 4 Amplification factor and phase error for Lax-Wendroff scheme ($r = 0.5$).

From Eq. (15), it can also be seen that the Lerat-Peyret generalized scheme has a lagging phase error for the low-frequency mode (notice the minus sign comes from the definition of ξ).

The analytical results give new insight into the influence of the various parameters on the dissipation and dispersion properties of each scheme. The plots of the exact expressions for the amplification factor and phase error reveal these characteristics in the previous figures.[6]

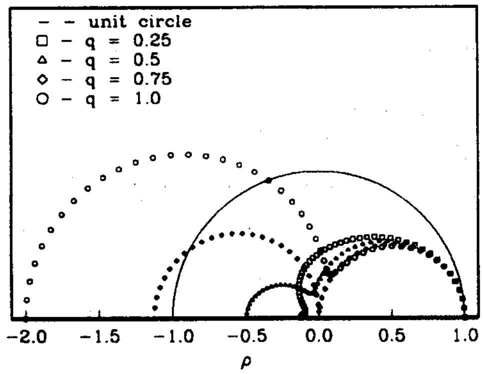


a) **Amplification Factor**

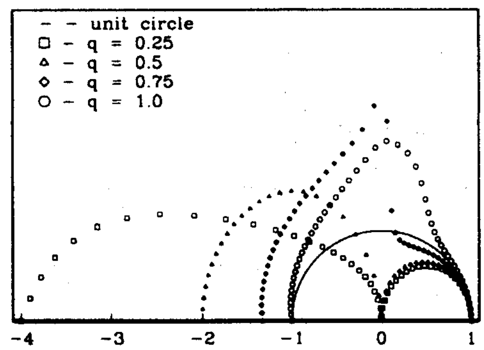


b) **Phase Error** Φ

Fig. 5 Amplification factor and phase error for Rubin-Burstein scheme ($r = 0.25$).



a) **Amplification Factor**

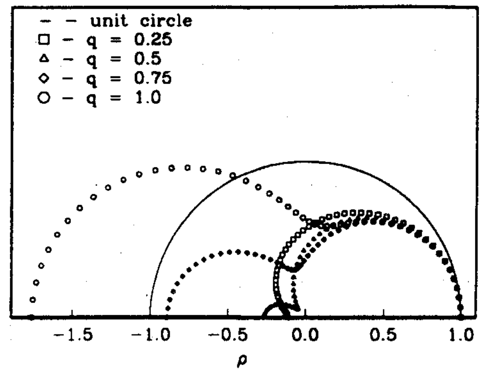


b) **Phase Error** Φ

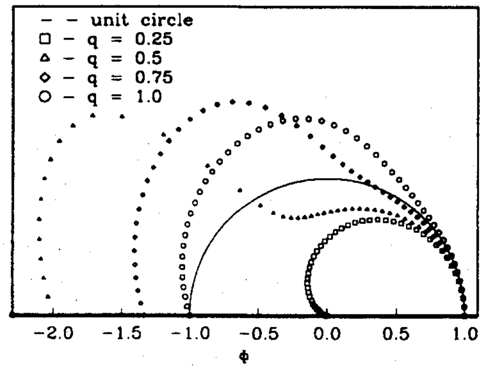
Fig. 6 Amplification factor and phase error for Rubin-Burstein scheme ($r = 0.5$).

Table 2 Amplification factor and phase error for small ξ

Burgers equation	Amplification factor
Inviscid	$\vert\rho(\xi)\vert = 1 - \frac{1}{8}q^2(1-q^2)\xi^4$
Viscous	$\vert\rho(\xi)\vert = 1 - r\xi^2 + \Psi_0\xi^4$
Burgers equation	**Phase error**
Inviscid	$\phi = -c\left[1 - \frac{1}{6}(1-q^2)\xi^2 + \left(\frac{1}{120} + \frac{1}{24}q^2 - \frac{1}{20}q^4\right)\xi^4\right]$
Viscous	$\phi = -c[1 - \frac{1}{6}(1-q^2)\xi^2 + \Gamma_0\xi^4]$



a) **Amplification Factor**



b) **Phase Error**

Fig. 7 Amplification factor and phase error for Peyret-Taylor scheme ($r = 0.25$).

4. *Stability Criteria*

Using the respective necessary and sufficient conditions for stability, the stability criteria for each scheme can be derived. This is neither an obvious nor easy task. Here a different approach is taken to obtain the stability criteria.

MacCormack scheme. From the graphs and surfaces of the amplification factor, it is observed that the maximum value of the amplification factor in the entire r, q region appears at ξ equal to π. Setting $\xi = \pi$, which means $a = 0$ and $b = 2$, we have

$$|1 - 2q^2 - 4(1-2r)r| \leq 1 \tag{16}$$

which leads to

$$-1 \leq 1 - 2q^2 - 4(1-2r)r \leq 1 \tag{17}$$

The right inequality is simplified to

$$q^2 + 2(1-2r)r \geq 0$$

which indicates limit on the diffusion parameter

$$r \leq ½ \tag{18}$$

Further investigation reveals that stability may be achieved for some values of q with $r > ½$.

The left inequality becomes

$$q^2 + 2(1-2r)r \leq 1$$

which means

$$q^2 \leq 1 - 2(1-2r)r \tag{19}$$

To verify this result, a numerical calculation was carried out as described in Ref. 6. From Table 3, it can be seen that condition (19) is slightly different from the numerical calculation, especially as r nears 0.5. It can also be seen from Fig. 2 that, when $r = 0.5$, at $\xi = 0.79\pi$, q must be less than 1 for $|\rho|^2 \leq 1$. This implies that there should be some type of higher order correction. Adding a higher order term of r^{12}, which is obtained by trial and error as the correction term and matching the numerical result at $r = 0.5$, we have

$$q^2 \leq 1 - 2(1-2r)r - 220.19r^{12} \tag{20}$$

Therefore, the stability criterion for the MacCormack scheme is

$$q^2 \leq 1 - 2(1-2r)r - 220.19r^{12}, \qquad r \leq ½ \tag{21}$$

The numerical calculation and conditions (19) and (20) are shown in Table 3 (in this table all data are for the equation $|\rho|^2 = 1$).

The commonly used stability criterion for the MacCormack scheme is the empirical formula of Tannehill et al.[11]:

$$q \leq 1 - 2r$$

The comparison of the exact stability criterion [conditions (19) and (20)] and the empirical formula is shown in Fig. 9, which indicates that the region for the stable solution is nearly double that for the empirical analysis.

Lax-Wendroff scheme. The Lax-Wendroff scheme is somewhat more complicated. From the graphs and surfaces of the amplification factor, when $r \leq 0.5$, the maximum value of the amplification factor appears at $\xi = 0$ or π. At $\xi = 0$, there is no meaning. So setting $\xi = \pi$, we have

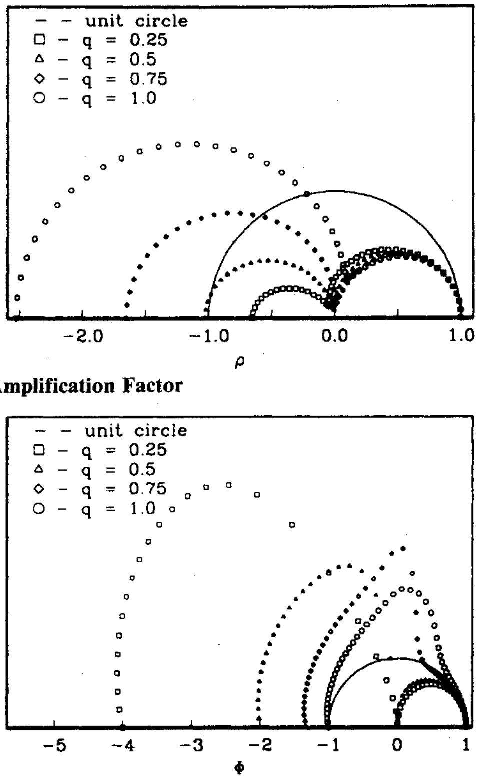
$$|1 - 2q^2| \leq 1 \tag{22}$$

which leads to

$$1 \geq q \geq 0 \tag{23}$$

Table 3 Numerical calculation and exact conditions (19) and (20) for stability for MacCormack scheme

	q		
r	Numerical	(19)	(20)
0.0	1.0	1.0	1.0
0.05	0.953939	0.953939	0.953939
0.1	0.916515	0.916515	0.916515
0.15	0.888819	0.888819	0.888819
0.2	0.871780	0.871780	0.871779
0.25	0.866025	0.866025	0.866018
0.3	0.871780	0.871780	0.871713
0.35	0.888819	0.888819	0.888401
0.4	0.916515	0.916515	0.914498
0.45	0.948039	0.953939	0.945948
0.5	0.972750	1.0	0.972750
0.50495	0.974804	0.0	0.0
0.505	0.0	—	—



a) **Amplification Factor**

b) **Phase Error**

Fig. 8 Amplification factor and phase error for Peyret-Taylor scheme ($r = 0.5$).

Actually, the right inequality is always satisfied. Thus this equation gives

$$q \leq 1 \tag{24}$$

When $r > 0.5$, the maximum value is at $\xi = 0.7\pi$. Upon substitution of this value, we have

$$\{0.6545q^2(1-3.1765r)^2 + [1 - 1.5878q^2 - 0.6545(1-1.5878r)r]^2\}^{½} \leq 1 \tag{25}$$

So the stability criterion for the Lax-Wendroff scheme is

$$\begin{cases} q \leq 1, & r \leq 0.5 \\ \{0.6545q^2(1-3.1765r)^2 + [1 - 1.5878q^2 \\ \quad - 0.6545(1-1.5878r)r]^2\}^{½} \leq 1, & r > 0.5 \end{cases} \tag{26}$$

The second condition, inequality (25), leads to

$$q \leq \sqrt{f(r) \pm 0.6552\sqrt{g(r)}} \tag{27}$$

where

$$f(r) = 0.5 + 0.4124r - 0.6552r^2$$

$$g(r) = 0.5823 + 2.170r - 3.4459r^2 - 0.002138r^3 + 0.002264r^4$$

Here we must have $g(r) \geq 0$, that is, $r \leq 0.8326$. Also the first inequality requires $r \geq 0.6296$ for $q^2 \leq 1$, and the second requires $r \geq 0.6298$ for $q^2 \geq 0$. This stable region is shown in Fig. 10.

Rubin-Burstein scheme. The maximum value of the amplification factor for the Rubin-Burstein scheme is at $\xi = \pi$. Using the same procedure, the stability criterion follows

$$|1 - 2q^2 - 2r| \leq 1 \tag{28}$$

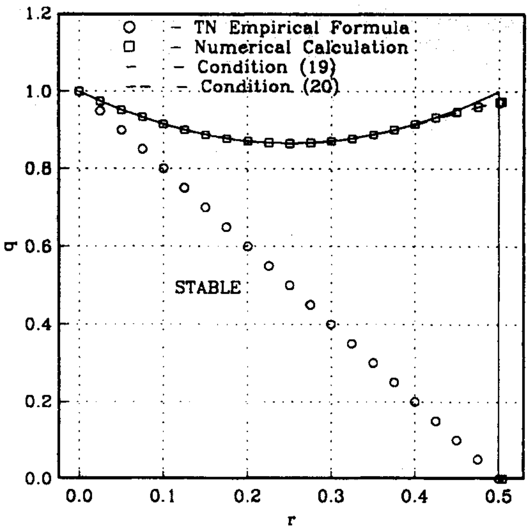


Fig. 9 Comparison of exact stability criterion and empirical formula for MacCormack scheme.

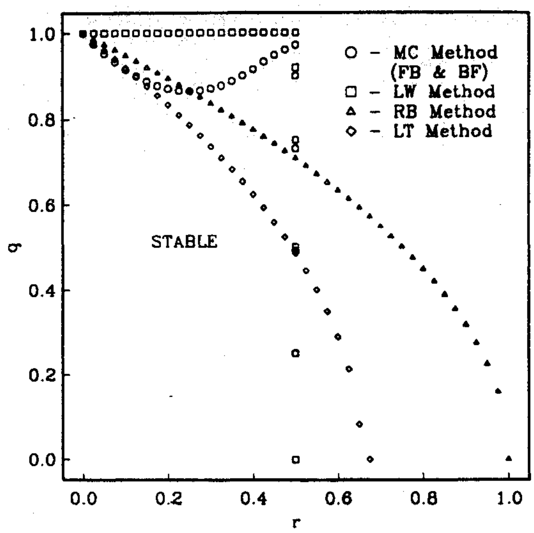


Fig. 11 Stable region for each scheme.

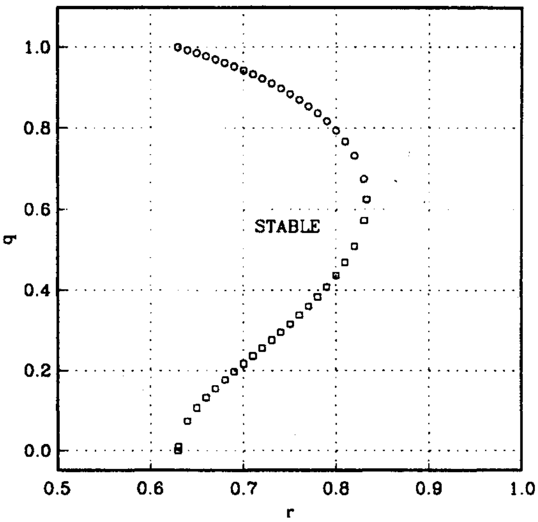


Fig. 10 Stable region for Lax-Wendroff scheme ($r > 0.5$).

which leads to

$$q^2 \leq 1 - r \tag{29}$$

Peyret-Taylor scheme. Similarly the stability criterion for the Peyret-Taylor scheme is

$$\left| 1 - 2q^2 - 4\frac{1+\sqrt{5}}{2+\sqrt{5}}r \right| \leq 1 \tag{30}$$

that is,

$$q^2 \leq 1 - 2\frac{1+\sqrt{5}}{2+\sqrt{5}}r \tag{31}$$

All of the previous results are shown in Fig. 11 and verified by numerical calculation. These stability criteria are also believed to be obtained here for the first time.

B. Discussion: Expansion to Navier-Stokes Equation

As stated earlier, the purpose of this study is to develop a stability analysis for the generalized two-level space-centered scheme applied to the Navier-Stokes equations. It has been shown that the Burgers equation has the same form as the one-dimensional Navier-Stokes equation. Therefore, in the analyses and procedure applied to the Burgers equation, the constants c and ν become the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, when the same analysis is applied to the one-dimensional Navier-Stokes equation and a local linearization is employed. The amplification factor becomes an amplification matrix. Now the task of stability analysis is to seek the maximum eigenvalue of this amplification matrix. From the illustrative example here, it has been demonstrated that such analyses are possible. The results shown here will provide meaningful and helpful guidance toward achieving such analyses as the next goal of this study.

V. Conclusions

According to the earlier results, the following conclusions can be obtained:

1) The von Neumann stability analysis has been performed for the generalized two-level space-centered scheme applied to the Burgers equation. The dissipation, dispersion, and stability properties of five different numerical schemes have been investigated. Generalized expressions for the amplification factor and phase error have been obtained and are presented in Eqs. (10) and (12).

2) The variation of specific parameters for each scheme is shown in Table 1. Initially the amplification factor and phase error were determined for the inviscid Burgers equation and were found to be identical for each scheme as expected. This verifies that, for the linear inviscid Burgers equation, the amplification factor and phase error are independent of α, β, and γ and all of these second-order accurate schemes have the same stability properties.[9,10]

3) The dissipation, dispersion, and stability properties of these five schemes are investigated for the linearized viscous Burgers equation for the first time. In the viscous case there are certain similarities among all five schemes, but each exhibits unique characteristics in its stability criteria. These features are evident in the plots in Figs. 1–8.

4) Of particular interest is the widely used MacCormack scheme for which the exact stability criterion almost doubles the stable region indicated by the empirical formula (see Fig. 9). This has significant practical meaning, because it extends the range of time and space step size and therefore improves the computational efficiency and accuracy.

5) It was also noted that for the MacCormack scheme, the order of the differencing (FB, forward predictor-backward corrector, or BF, backward predictor-forward corrector) gave the same dispersion property in each case. Thus, for the linear case, alternating the differencing in successive steps does not reduce the phase error.

6) For the Lax-Wendroff scheme there exists another stable region when $r > 0.5$ (see Fig. 10).

This study is continuing in an effort to determine the physical significance of these features as well as to identify any additional properties that affect stability. Although this study has been directed toward the linearized viscous Burgers equation, it has provided valuable understanding that can be extended to the nonlinear viscous Burgers equation. This, in turn, will serve as a guide to the analysis of critical stability and accuracy issues in a wide class of finite difference schemes for the complete Navier-Stokes equations.

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References

[1]Warming, R. F., and Hyett, B. J., "The Modified Equation Approach to the Stability and Accuracy Analysis of Finite-Difference Methods," *Journal of Computational Physics,* Vol. 14, No. 2, 1975, pp. 159–179.

[2]Hirt, C. W., "Heuristic Stability Theory for Finite-Difference Equations," *Journal of Computational Physics,* Vol. 2, No. 4, 1968, pp. 339–355.

[3]Griffiths, D. F., Christie, I., and Mitchell, A. R., "Analysis of Error Growth for Explicit Difference Schemes in Conduction-Convection Problems," *International Journal for Numerical Methods in Engineering,* Vol. 15, No. 7, 1980, pp. 1075–1081.

[4]Hindmarsh, A. C., Gresho, P. M., and Griffiths, D. F., "The Stability of Explicit Euler Time-Integration for Certain Finite Difference Approximations of the Multi-Dimensional Advection-Diffusion Equation," *International Journal for Numerical Methods in Fluids,* Vol. 4, No. 4, 1984, pp. 853–897.

[5]Klopfer, G. H., and McRae, D. S., "The Nonlinear Modified Equation Approach to Analyzing Finite Difference Schemes," AIAA Paper 81-1029, June 1981.

[6]Yao, Y.-P., "Stability Analysis of Generalized Two-Level Space-Centered Difference Schemes for the Convection-Diffusion Equation," Master's Thesis, Ohio State Univ., Columbus, OH, June 1991.

[7]Sod, G. A., *Numerical Methods in Fluid Dynamics—Initial and Initial Boundary-Value Problem,* Cambridge Univ. Press, 1985.

[8]Peyret, R., and Taylor, T. D., *Computational Methods for Fluid Flow,* Springer Series in Computational Physics, Springer-Verlag, Berlin, 1983.

[9]Anderson, D. A., Tannehill, J. C., and Pletcher, R. H., *Computational Fluid Mechanics and Heat Transfer,* Hemisphere, 1984.

[10]Lerat, A., and Peyret, R., "The Problem of Spurious Oscillations in the Numerical Solution of the Equations of Gas Dynamics," *Lecture Notes in Physics,* Vol. 35, Springer-Verlag, Berlin, 1975, pp. 251–256.

[11]Tannehill, J. C., Holst, T. L., and Raskich, J. V., "Numerical Computation of Two-Dimensional Viscous Blunt Body Flows with an Impinging Shock," *AIAA Journal,* Vol. 14, No. 2, 1976, pp. 204–211.